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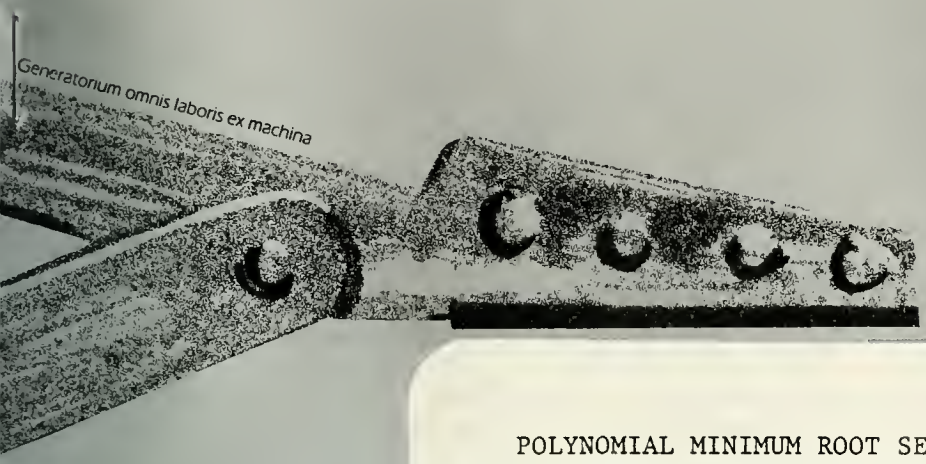

POLYNOMIAL MINIMUM ROOT SEPARATION
(NOTE TO A PAPER OF S.M. RUMP)

By

J.T. Schwartz

Technical Report No. 155
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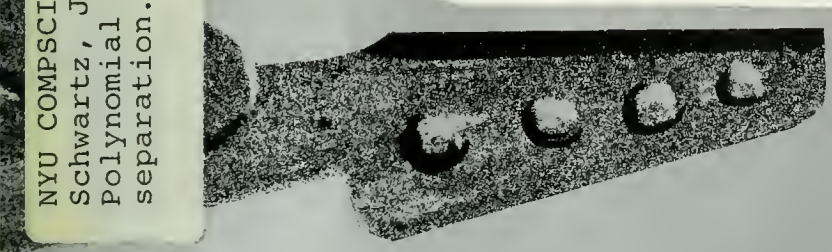

New York University
Courant Institute of Mathematical Sciences

Computer Science Division
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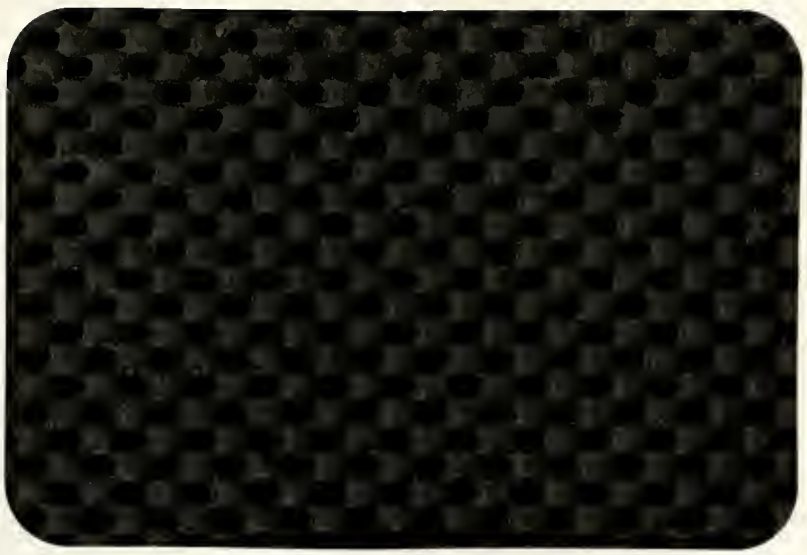

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C. 1

Let P be a polynomial of degree d, and let its distinct zeroes be $z_1, \ldots, z_n$, the integers $m_1, \ldots, m_n$ being their respective multiplicities. Then the logarithmic derivative $L(z)=P'(z)/P(z)$ can be written as

$$L(z)=P'(z)/P(z) = \sum \frac{m_l}{z-z_l}. \qquad (1)$$

If r and θ respectively designate the modulus and argument of $z-z_l$, then $-(\bar{z}-\bar{z}_l)^{-1}=-r^{-1}e^{i\theta}$, and hence $\overline{-L(z)}$ can be interpreted as the total force in the complex plane exerted by masses m_l at the points z_l, where the law of attraction is inverse distance (see [M49]). The equilibria of this field of force are the points z satisfying $P'(z)=0, P(z)\neq 0$ that we seek. Because of this, we shall refer to such z as ezeroes of P'.

Definition 1: A smooth simple closed curve C in the complex plane which does not pass through any zero of P or P' will be called **attracting** for P if the component of the force $\overline{-L}$ in the direction of the inward normal to C is always positive.

If the curve C is attracting, then along C the force $\overline{-L}$ can clearly be deformed homotopically into the normal vector to C without ever passing through zero; hence as we traverse C, $-L$ turns through the same total angle as does the normal (or equivalently, the tangent) to C. This is of course 2π. But as L traverses a curve about zero homotopically equivalent to $e^{i\theta(t)}$ its complex conjugate $\bar{L}$ traverses a curve homotopically equivalent to $e^{-i\theta(t)}$. Thus the net turning of L is -2π, i.e. is equivalent to that of z^{-1}, which is to say that of a single pole. Hence the following Lemma is an immediate consequence of Rouche's theorem:

Lemma 2: If P is a polynomial as above and C is simple closed attracting curve for P which contains exactly k distinct zeroes of P, then C contains a total of $k-1$ ezeroes of P'. (Counted by multiplicity; but these ezeroes need not be distinct.)

Our next aim is to show that any pair of distinct zeroes z_1, z_2 of P are surrounded by an attracting circle C lying at a distance from $(z_1+z_2)/2$ whose radius can be bounded by a function of the degree of P and the number of distinct zeroes of P which C contains. It will be convenient for this to take $z_1=-1$, $z_2=+1$. The following Lemma estimates the minimum 'attractive force' that these two points exert on a point lying on a circle of radius R.

Lemma 3: If $R>1$, then as z varies over the circle C of radius R, the minimum normal component of the 'force' $(z-1)^{-1}+(z+1)^{-1}$ is attained at $z=+R$, and is simply $2R/(R^2+1)$.

Proof: This normal component is just

$$\mathrm{Re}\left(\frac{z}{|z|}\left((z-1)^{-1}+(z+1)^{-1}\right)\right)=\frac{1}{|z|}\,\mathrm{Re}\left(\frac{2z^2}{z^2-1}\right)=\frac{2}{|z|}\,\mathrm{Re}\left(1+\frac{1}{z^2-1}\right)=\frac{2}{|z|}\left(1+\mathrm{Re}\left(\frac{1}{z^2-1}\right)\right)$$

As z varies over the circle of radius R, $\zeta=z^2=R^2e^{2i\theta}$ varies twice over the circle C of radius R^2, and hence $\left(\zeta-1\right)^{-1}$ varies over the image C'' of C' under $\zeta \to \left(\zeta-1\right)^{-1}$. This is the circle symmetric to the real axis through $(R^2-1)^{-1}$ and $-(R^2+1)^{-1}$, and this latter value plainly represents the largest negative value of $\mathrm{Re}(z^2-1)^{-1}$. Hence the desired minimum is $2R^{-1}\left(1-(R^2+1)^{-1}\right)=2R(R^2+1)^{-1}$. Q.E.D.

Lemma 4: For each $y\geq 0$, let C_y designate the circle with radius y and center at the origin. Suppose that the polynomial P has roots at $z=1$, $z=-1$, that $R>1$, that C_R passes through none of the roots of P, but that C_R is not attracting for P. Then there exists an integer $k\geq 1$ such that C_{R3^k} contains at least k zeroes of P that do not belong to C_R.

Proof: For each $x \geq R$, let $N_R(x)$ designate the number (counted with multiplicity) of zeroes of P that lie in the annulus C_x-C_R, and suppose that the assertion of the Lemma is false, i.e. that $N_R(R3^k)<k$ for every integer $k\geq 1$, so that in particular every zero of P in $|z|\geq R$ satisfies $|z|\geq 3R$. Since the circle C_R is not attracting, there exists a point z on it such that

$$\sum_{|z_l|\geq R}(z-z_l)^{-1}$$

has a larger component in the outward normal direction then the minimum inward normal force exerted by the two roots $+1,-1$ of P. By the preceeding Lemma, this gives

$$\mathrm{Re}\left(\sum_{|z_l|\geq R}\frac{e^{i\theta}}{\mathrm{Re}^{i\theta}-z_l}\right)\geq\frac{2R}{1+R^2}, \tag{2}$$

where $z=\mathrm{Re}^{i\theta}$. Since $\mathrm{Re}\,(R-ze^{i\theta})^{-1}\leq(|z_l|-R)^{-1}$, (2) gives

$$\frac{2R}{1+R^2}\leq\sum_{|z_l|\geq 3R}(|z_l|-R)^{-1}=\int_{3R}^{\infty}\frac{1}{x-R}dN_R(x) \tag{3}$$

$$=\int_{3R}^{\infty}\frac{N_R(x)}{(x-R)^2}dx.$$

On the other hand, since $N_R(R3^x)\leq N_R(R3^k)\leq k-1\leq x$, where k is the smallest integer not less than x, we have $N_R(Ry)\leq\log_3 y$, so $N_R(y)\leq\log(y/R)/\log 3$. Putting $f(t)=\log(y/R)/\log 3$, it follows from (3) that

$$\frac{2R}{1+R^2}\leq\int_{3R}^{\infty}\frac{f(x)}{(x-R)^2}dx=-\frac{f(x)}{x-R}\Big|_{3R}^{\infty}+\int_{3R}^{\infty}\frac{f'(x)}{x-R}dx \tag{4}$$

$$=\frac{1}{2R}+\frac{1}{\log 3}\int_{3R}^{\infty}\frac{dx}{(x-R)x}\leq\frac{1}{2R}\left(1+\frac{1}{\log 3}\right)<\frac{1}{R},$$

implying that $2R^2<1+R^2$, which is impossible since $R>1$. Q.E.D.

Corollary 5: Let P, R, C_γ etc. be as in the preceeding Lemma. Then there exists an integer K such that C_{R3^K} is attracting for P and contains at least K zeroes of P that do not belong to C_R.

Proof: By the Lemma, there exists an integer $k \geq 1$ such that C_{R3^k} contains at least k zeroes of P that do not belong to C_R. If C_{R3^k} is attracting, the present Corollary is proved. Otherwise we can apply the Lemma to this circle, giving an integer j such that $C_{R3^{k+j}}$ contains at leaset $k+j$ zeroes of P that do not belong to C_R. Iterating this step sufficiently often must clearly produce a K satisfying the Corollary, since a large enough circle contains all roots and is therefore attracting. Q.E.D.

Lemma 6: Let P be a polynomial of degree d having roots at $z = 1, z = -1$. Then either the circle $|z| \leq 1$ contains an eroot of $P'(z)$ or the circle $|z| \leq 3^{d-2}$ contains at least two eroots of $P'(z)$ (counted with multiplicity.)

Proof: Take a small positive ϵ such that the preceeding Corollary applies to the circle $C_{1+\epsilon}$. if this circle is attracting for P, then by Lemma 2 it contains at least one ezero of P' and the present Lemma is proved. Otherwise the preceeding Corollary implies the existence of a $k \leq d-2$ such that $C_{(1+\epsilon)3^k}$ is attracting and contains at least 3 distinct roots of P, and therefore at least two eroots of P' (counted with multiplicity.) It follows that $C_{(1+\epsilon)3^{d-2}}$ contains at least this many eroots of P'. Q.E.D.

Corollary 7: If P is a polynomial of degree d having distinct roots z_1, z_2, then either the closed circle with z_1, z_2 contains an eroot of P', or the circle with center $(z_1+z_2)/2$ and radius $3^{d-2}|z_1-z_2|$ contains two eroots of P' (counted with multiplicity.)

Proof: Obvious from the preceeding Lemma.

The remainder of our argument follows that of [Ru79] quite closely. More specifically, we will prove:

Theorem 8: Let P be an arbitrary polynomial of degree d with integer coefficients. Then the minimum distance $sep(P)$ between distinct roots z_1, z_2 of P satisfies

$$sep(P) \geq \left(2ed^{(d+3)/2}(|P|+1)^d\right)^{-1}.$$

For this we can use the following

Lemma 9: Let P and Q be arbitrary polynomials with integer coefficients having degrees d, d^*. Let $z_1 \cdots z_d$ be the roots of P (counted with multiplicity) and let $q_1, \ldots, q_a$ be the moduli of those nonzero values $Q(z_1), \ldots, Q(z_a)$ which lie inside the unit circle. Then

$$q_1 \cdots q_a \geq \left(d^*(|Q|+1)^d P^{d^*}\right)^{-1}.$$

Proof: Assume the contrary and let z_1, z_2 be roots violating the inequality stated. Since P and $z^d P(z^{-1})$ have the same norm $|P|$, we can plainly assume that one of these roots, say z_1, satisfies $|z_1| \le 1$. Let $\delta = |z_1 - z_2|$. By corollary 7, there is either (1) an eroot ζ of P' within distance 2δ of z_1, or (2) a double eroot ζ of P' within distance $2^{d-1}\delta$ of z_1, or (3) two distinct eroots ζ_1, ζ_2 of P' within distance $2^{d-1}\delta$ of z_1. First consider case (1). Any smooth complex function of a real variable x satisfies

$$f(h) = f(0) + hf(0) + \int_0^h (h-x) f''(x) dx,$$

so

$$|f(1)| \ge |f(0)| - |f'(0)| - \frac{1}{2} \max_{0 \le h \le 1} |f''(R)|. \tag{5}$$

Applying this inequality to $P\left((z_1 - \zeta) + \zeta\right)$ gives $|P(\zeta)| \le 2\delta^2 M$, where M is the maximum of P'' on the segment connecting z_1 and ζ. Since $|z_1| \le 1$, we have plainly $M \le d^2 |P| (1+\delta)^{d-2}$, and so $|P(\zeta)| \le 2d^2 \delta^2 (1+\delta)^{d-2} |P|$. In case (2) we can use the inequality

$$|f(1)| = |f(0)| - |f'(0)| - \frac{1}{2}|f''(0)| - \frac{1}{6} \max_{0 \le h \le 1} |f'''(h)|, \tag{6}$$

which can be derived in the same way as (5), to prove in similar fashion that

$$|P(\zeta)| \le \frac{1}{6} d^3 (1+\delta)^{d-3} 2^{3(d-1)} \delta^3 |P|. \tag{7}$$

In case (3), we can obtain

$$|P(\zeta_1) P(\zeta_2)| \le 4d^4 2^{2(d-1)} \delta^4 (1+\delta)^{2d-4} |P|^2 \tag{8}$$

by multiplying two inequalities like that derived in case (7).

Applying Lemma 9 we therefore obtain the following inequalities:

In case (1): $2d^{(d+3)/2}(|P|+1)^d \ge \delta^{-1}(1+\delta)^{-(d-2)/2}$

In case (2): $2^{d-1} d^{(d+4)/3}(|P|+1)^{2d/3} \ge \delta^{-1}(1+\delta)^{-(d-3)/3}$

In case (3): $2^{d/2} d^{(d+5)/4}(|P|+1)^{(2d+1)/4} \ge \delta^{-1}(1+\delta)^{-(d-2)/2}$.

The case (1) inequality is always weaker than the case (3) inequality, since always $2^{1-d/2}(|P|+1)^{d/2-1/4} \ge 1$. Moreover, Theorem 8 can be verified by elementary means if either $d \le 2$ or $|P| \le 2$. But if $d \ge 3$ and $|P| \ge 3$, the case (1) inequality is weaker than the case (2) inequality, since

$$2^{2-d} d^{(d+1)/6}(|P|+1)^{d/3}(1+\delta)^{1/6} \ge 2^{2-d/3} d^{d/6} \ge 1.$$

Hence the case 1 inequality can be asserted in all nontrivial cases, from which we can conclude that $\delta \ge \left(ed^{(d+3)/2}(|P|+1)^d\right)^{-1}$, since if this is false we plainly have $\delta \le 2/(d-2)$, so that $(1+\delta)^{(d-2)/2} \le e$. Q.E.D.

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